THE BEGINNINGS OF SET THEORY

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Fellow Services Office, early in the academic year 2005/06.
Itay Neeman: Darf ich Englisch sprechen? Ich verstehe nur wenig Deutsch.
Martin Garstecki: Yes of course! Come in. Have a seat. What can I do for you?
Itay Neeman: I was wondering if you could help me find something.
Martin Garstecki: Certainly. What are you looking for?
Itay Neeman: I'm looking for the Truth.
Martin Garstecki (without so much as a blink of the eye): The Truth. Normally Herr Meyer-Kalkus would be responsible for this. But just today he is not here at the Wissenschaftskolleg. (Pause.) Let me see … Perhaps Frau Bottomley. Let's see …
Martin Garstecki (to telephone): Hallo Frau Bottomley, Martin Garstecki hier. Gut, danke, und Ihnen? Ich habe hier im Büro Herrn Neeman. Er will die Wahrheit finden. (Pause.)

Ja, ich weiß, aber Herr Meyer-Kalkus ist heute nicht im Wissenschaftskolleg. (Pause.) Dann schicke ich ihn zu Ihnen. Tschüss.

Martin Garstecki (to Itay Neeman): Good news. Frau Bottomley can help you. Do you know how to get to the library? You just go down one floor and it's right there.

Itay Neeman: Thank you so much. Tschüss.

A few minutes later, down at the library.

Itay Neeman: Ah, hello, I'm looking for Gesine Bottomley.

Gesine Bottomley: Oh, you must be Herr Neeman. Here, I have something for you. (Hands Truth over to Itay Neeman.)

Itay Neeman (inspects Truth, says): This is exactly what I was looking for! Thank you so much!

Now, this did not actually happen to me at the Wissenschaftskolleg, but only because I did not ask. Having spent a year as a Fellow, I am convinced that the members of the staff have everything, Truth included, if not in one of the offices, the library, or the kitchen, then at the very least available for rent from Herr Riedel, in case a Fellow ever needs it. The Wissenschaftskolleg is one of those rare places where academics, for better or for worse, are treated like royalty. And who can resist that?

My year at the Wissenschaftskolleg started very briskly. I knew of course that everyone is afraid of mathematicians. (Only dentists and tax inspectors surpass mathematicians in this respect, and they were not present at the Wissenschaftskolleg.) So I decided to give my talk early in the year, and drop the standard format of an hour's talk followed by an hour of questions and answers, in favour of a more informal mixing, taking questions as they come up. It was an interesting experience, for me at least and I hope also for my colleagues.

My subject, set theory, was born in 1874 with the publication of Georg Cantor's paper *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen* demonstrating that there are more real numbers than there are natural numbers.

The natural numbers (a mathematical jargon, not meant to imply any deviance on the part of other numbers) include zero, one, two, three, and so on. The real numbers include also twelve and three-sevenths, the square root of two, pi, and many more. Of course, you say, there are more real numbers than there are integers. But before jumping to conclusions let us look at the integer numbers, both positive such as zero, one, two, etc., and negative, such as minus one, minus two, etc. It is tempting to think that there is a greater quantity of integer numbers (why, twice as many!) than natural numbers. But this is an illusion, brought about by the *order* of the *presentation* of the two sets. We tend to think of the naturals as stretching from zero to plus infinity, and of the integers as stretching from minus infinity, through zero, to plus infinity. It is this ordering that we have in our minds that deceives us into the hasty conclusion that the quantities are different. We are not used to being wary of the order of presentation, since in the case of finite sets it does not interfere with our perception of quantity. But in the case of infinite sets we have to be more careful. If we were to change the order of the integers, and count them as zero, minus one, one, minus two, two, etc., then we could place them directly above the naturals, with zero above zero, minus one above one, one above two, minus two above three, two above four, minus three above five, three above six, and so on. We would then see that in fact *both* sets fit on the ray of naturals from zero to plus infinity. More precisely we would obtain an exact correspondence between naturals and integers, with each integer corresponding to exactly one natural, and each natural corresponding to exactly one integer.

With our perception of quantity endangered by the vagaries of presentation in the case of infinite sets, we require a precise definition. Reflecting, we see that we think of two sets as having the same quantity if *the elements of one can be placed in exact correspondence with the elements of the other*. This definition abstracts our notion of quantity, striping away the dangers of specific presentations using the word *can*. If there is *any* way of placing the elements of one set in exact correspondence with the elements of the other, then the two sets are of the same quantity. The original presentation does not matter.

Equipped with an abstract definition, we can approach the question of quantities for infinite sets. We see now that the set of integers and the set of naturals in fact have the same quantity, since there *is* a way, perhaps not the way we imagine them at first, but still there *is* a way, to put them in exact correspondence.

Next we think of the rational numbers, namely all fractions with integer nominator and denominator, such as one half, fifty seven over thirteen, and so on. Surely there are more rational numbers than there are natural numbers? Well, no. Again it is just an illusion brought about by the order of the numbers in our minds. There is a way, and I explained it in my talk but will spare you the details now, to place the rational numbers in exact correspondence with the natural numbers.

By now you must be thinking that any two infinite sets are of the same quantity. That if you take any two infinite sets, and just order them in the right way, then you will see that they can be placed in exact correspondence. And here it is that Cantor's paper provides a great surprise. Cantor showed that the set of real numbers, which remember include such beauties as the square root of two, pi, and indeed all limits of bounded increasing sequences of rational numbers, is *not* of the same quantity as the set of natural numbers. Just to give you a glimpse of the magnitude of the discovery let me emphasize: he showed that there is *no way*, no matter how hard you try, to put the real numbers and the natural numbers in an exact correspondence.

How could he go over all potential ways of setting an exact correspondence (why, surely there are at least infinitely many!) and show that none of them works? By abstract reasoning: assuming the existence of an exact correspondence, not knowing what it is, but still reasoning about properties it must satisfy, properties that follow just from the fact that it is an exact correspondence between the real numbers and the natural numbers, until reaching two contradictory properties, thereby demonstrating that a correspondence of this kind cannot exist.

It is this process of reasoning, in mathematical jargon a *proof*, that is the bread and butter of a mathematician's day (and night, more often than not). In my talk I wanted to illustrate this. I wanted to give my colleagues at the Wissenschaftskolleg a sense of this method of proof, central to the very existence of pure mathematics. And so I presented Cantor's proof (not the one in the paper cited above, but a later one, the extremely elegant *diagonal argument*, that he found in 1877). It must have been quite a shock to the audience. The academic year had only started a few weeks earlier, we were mostly strangers to each other still, and, let's be blunt, a mathematical proof is not foremost on the mind of, well, anyone who is not a mathematician. But it was a fantastic experience. The questions during the talk came from people eager to understand the concepts, the reasoning, the methods, and the goals. Such was the interest, my colleagues' willingness and even desire to probe into the mindset of a mathematician, that two hours barely seemed sufficient. It was a very nice beginning for a very nice year.

Cantor's proof initiated set theory, the study of the world of sets, particularly infinite sets. The field has of course changed a great deal since. It has made extensive use of logic, which allows formalizing the concept of proof, and applying the same powers of reason that Cantor used on exact correspondences, to proofs. In other words, not only can we prove things, but we can also prove that certain things cannot be proved, namely that some proofs, just like exact correspondences between the reals and the naturals, do not exist. Thus our knowledge of the universe of sets, which can only come through proofs that do exist, is inherently incomplete. Gödel discovered this in one form, and later on Cohen discovered a complementary form.

Further, and this is getting closer to my own work, it has been discovered that there is a natural hierarchy of axioms about the universe of sets, asserting in a sense that there are truth-preserving embeddings acting on it. These axioms appear a priori to be far removed from such concrete objects as the real numbers. But surprisingly it turned out that these axioms do affect properties of real numbers. (My colleague John Steel is one of the people responsible for the central developments in this line of research.) The intermediary connecting the strong axioms of set theory to concrete properties of real numbers is a principle known as determinacy, asserting the existence of winning strategies in definable infinite games.

My own research centres on determinacy, in other words on the existence and applications of winning strategies in infinite games, with the set theoretic axioms on one hand, and definable sets of real numbers on the other. Working at the Wissenschaftskolleg, I completed several articles on this, and initiated a few more, of which two have so far been completed. I was fortunate to have the help of many visitors, from the US, China and Israel, colleagues in Germany and of course John Steel, with whom I shared an office. I was also fortunate to have the company of good friends and colleagues outside mathematics with many interesting stories to tell. It has been a wonderful year, and I only wish we could all do it again.