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Symmetry in Chaos*

Symmetry in Nature was the title of a talk that my friend, the well-known physicist and group theoretician Marcos Moshinsky, has held before quite different audiences and in maybe fifty different versions. I have heard his exposition on more than one occasion and never heard the same talk twice. The field is wide and rich and can be connected to fields that range from mathematics over the natural sciences and all the way down to the fine arts.

I would have liked to follow in the steps of my teacher but I felt overwhelmed by the wealth of the field. As our work this year is centered around the concept of chaos, I considered that it might be appropriate to discuss how symmetry, this supreme concept of order, may interplay with chaos, which we intuitively see as the very essence of disorder. I shall try to give an idea how two concepts of symmetry, the conventional one and a new structural one intermesh in a way that will be familiar to the expert from other contexts. What I plan to present is certainly not acceptable for any specialized journal. Nevertheless it will present both new ideas and new views of old ideas that are at the basis of some of my recent research papers, and I hope that some of the basic thought behind this work may make sense to the non-specialist.

I wish to recall that we distinguish — maybe somewhat artificially - between disordered systems and systems with dynamical chaos.

In the first case, we actually assume that the object we study is affected by a statistically disordered environment, such as, for example, the scattering of light or of massive particles, such as neutrons, off a crystal structure with statistically distributed impurities or off amorphous substances where order is totally absent such as gases, liquids or glasses.

In the second case we consider a usually quite simple, but non-linear system which evolves according to a well defined deterministic non-statistical law, but where this evolution turns out to be highly unstable and sensitive to initial conditions.

Note that both things can occur simultaneously and are not always well distinguished. Thus, presumably, the chaotic movements of the molecules in a gas are due to the chaotic dynamics of the system, but the scattering of

* Colloquium presented at the Wissenschaftskolleg, July 20, 1993.

particles on such a gas will be viewed as a process happening on a disordered medium.

Symmetry in chaotic systems has received very little attention, and such attention was mostly restricted to small discrete symmetry groups such as reflection, time reversal, permutations and finite rotations. These symmetries are of considerable importance and the time reversal invariance has given rise to some very deep results in the field of quantum chaos. Yet I want to address a completely different concept of symmetry that I used in some recent work and that can best be understood if we try to view the similarities of disorder and chaos rather than the differences.

Consider that one of the very first assumptions we make in any disordered system is isotropy and homogeneity. This implies for amorphous systems a very strong symmetry, namely invariance under rotations and translations, i.e. under the full Euclidean group. How can this be, as we just introduced disorder to destroy the symmetry? Here we deal with invariance in a statistical sense. The individual sample will certainly not have symmetry; but if we consider an ensemble, this ensemble will be invariant. We expect such a situation for any amorphous substances, such as gases, liquids or glasses etc. In the cases of gases or liquids, i.e. when we are not dealing with solids, we may also view this large symmetry as a time averaged symmetry for a specific system rather than an ensemble property.

You have been exposed to an important example of such considerations in the multiple discussions of mesoscopic systems that were presented here. In the simplest case, namely the one-dimensional situation, the impurities were always assumed to be randomly distributed along the wire, and only the leads (surface) gave rise to a breaking of this symmetry. Based on that, statistics were made with an ensemble of impurity distributions that is invariant under translations along the wire while each individual member of the ensemble does not have this property.

Clearly, the physical consequences of a statistical symmetry are not the same as those of a symmetry of the individual sample. On the contrary, in the example of one-dimensional wires symmetry of the individual sample leads to extended states while statistical invariance leads to localization; in a loose way we might say that the two kinds of symmetry are complementary as they have opposite effects. In general we shall proceed to show that the two concepts of symmetry are indeed *complementary* in a sense that is mathematically well defined and indeed not uncommon in applications of group theory.

The concept of complementarity is defined as follows: The pair of subgroups q, $\subset$ 9 is complementary in the Group 9 if their elements commute and if neither of the two can be enlarged and still fulfill the commutation condition. This means in simple terms that if 9 is the largest group of transformations contemplated for our system and if, say, ' is the actual symmetry group of this system, then R' is the group of all those transformations we can do without violating the symmetry.

If we have no further information than the symmetry of a system, it is plausible to admit all transformations that conserve this symmetry in order to inspect the universe of systems that have this symmetry. This will naturally form the ensemble which we wish to study if we intend to extract some statistical information. This seems very fruitful and almost an obvious path in the light of usual information theoretic approaches. One relevant question remains: Complementary in what larger group? We shall see that once answered this question, this concept will be very fruitful in determining the ensembles which we ought to study in a given case.

Let us now turn our attention to chaotic, but deterministic systems. Among them we shall focus on so-called kicked systems as these are easiest to consider in analogy with disordered systems. In particular let us look at the kicked rotor. It is defined as a two-dimensional rotor which receives periodically a kick that is a delta function in time and some function of the angle. Such a system has actually been shown to be equivalent statistically to a banded random matrix problem, with a band shape dependent on the function of the angle. We can now see what happens in phase space, which consists of an angle, and the corresponding angular momentum, which can extend to infinity. In this phase space we may in general find ordered and disordered structures. In particular some invariant torus may block the path to infinite angular momentum and thus give compact chaotic pieces of phase space. It is thus a good example for a periodically time-dependent system, and has been widely used as a paradigm. The only non-typical feature stems from the low dimensionality that precludes a phenomenon called Arnold diffusion; indeed in higher dimensions typically all chaotic regions are connected.

Having the above example in mind we now study a general periodic one-dimensional system represented by a Hamiltonian $H(q, p) = H(q, p, t + T)$, where T is the period of the system. We can then consider the time evolution over a period which is a map of the space coordinate q and the momentum p at time t onto those at time $t + T$. We consider such a map to be chaotic if the iteration scatters points all over the (q, p) space or phase space that is available. Such a phase space can be described in different coordinate systems, but we must maintain that the two coordinates behave with respect to each other like a space coordinate and a momentum. This is guaranteed by admitting canonical transformations only. On the other hand, the map of time evolution itself must be canonical for the same reason. It is therefore tempting to consider the group of all (invertible) canonical transformations $\mathcal{G}$ as the large group in which we will have to set the scheme of complementarity outlined above. A completely chaotic system with no symmetries whatsoever will be statistically invariant under all transformations of this large group $\mathcal{G}$ in the sense that time averages will not be affected. Any symmetry, continuous or discrete will be broken by some transformations of this group. Therefore we shall have statistical invariance with respect to such transformation only, which commute with the elements of the symmetry group. The symmetry group itself has to be a subgroup of $\mathcal{G}$. The group of all transformations from $\mathcal{G}$ that commute with the symmetry group is thus complementary in $\mathcal{G}$ to the symmetry group.

Indeed this is exactly the structural invariance group defined in [1], without the use of the concept of complementarity. There it was put to use for a general proof of the connection between spectral statistics and chaos, which, up to now, is the main result of our work in this context. As this proof is presented elsewhere and is also quite technical, we shall skip it here and pass to a particular aspect not previously treated. In the proof, strong emphasis was put on the limitation to compact phase spaces, i. e. phase spaces that do not extend to infinity in either momenta or space coordinates. Sound technical reasons exist for this limitation, but the physical background is more interesting:

It is diffusion. A particle will take some time in an extended system to get from one part of the system to another however chaotic its motion may be. This really means that transformations, that will take parts of configuration space into each other, that are very far apart, will certainly not leave invariant any properties on a time scale that is not large compared to the time needed to get from one such part to another. Something similar will hold for momenta if arbitrary large transfers of momenta per unit time are not allowed. Thus we have to keep track of the time scale on which diffusion takes place. This time scale will depend on the energy available and thus for any finite phase space there will be an energy high enough that the time we are interested in is large compared to the diffusion time. If the space is infinite, on the other hand, this is no longer true. We can also say in the terms used above, that in such cases we have to take into account another essential piece of information that is not a symmetry.

If we are interested in quantum phenomena, another time scale enters the game; this scale relates directly to quantum phenomena and is often called the break time and at times beyond this time analogies with classical mechanics break down completely. As we go to very high energies we can approach the classical limit and the break time typically goes to infinity. Thus according to the results obtained in [1] for compact systems, the connection between chaos and random-matrix models associated to the classical ensembles [2] hold as is also suggested by a theorem of Schnirelmann

[3], who shows that wave functions are ergodic in similar cases. On the other hand if the system is infinite, this clearly is no longer true as diffusion time will also be infinite. In this case the invariance holds locally, but globally it fails. In practice this is even more stringent because we may not be near enough to the classical limit for an extended but finite system to explore all its phase space and thus such situations frequently occur.

This problem is often taken into account by breaking the extended system into sufficiently small pieces for diffusion time to be faster then any characteristic time of the system except the free flight transit time, yet large enough to be compared to the mean free flight path in the system. For these small pieces some random matrix ensemble is usually assumed. Examples of such a procedure are the treatments of mesoscopic systems by H. Weidenmüller or P. Mello [4] and their respective collaborators.

Another approach lies in the use of more particular random matrix ensembles such as banded matrices, which in the case of the kicked rotator give exactly the right answer [5]. This leads us to an important phenomenon, namely localization. Banded matrices with exponentially decaying bandwidth can indeed be shown to display Anderson type localization [6]. Thus states of such a system are not extended, even if the classical motion goes off to infinity.

If, on the other hand, we look at a system with translation symmetry individually (as opposed to such a symmetry for a time average or an ensemble), eigen-states will always be extended and time evolution will extend to infinity. This will always be true even if the classical motion is confined, e. g. by potential periodic barriers.

The essential difference between the two complementary concepts of symmetry (or invariance of individual systems) on one hand and of structural (or statistical) invariance on the other is thus very apparent in this asymptotic domain of large time behaviour. A fruitful exploitation of these ideas beyond the result of [1] will depend on a number of points. First, a better understanding of the group in which the complementary groups are embedded. Second, an understanding of their relation to dynamical groups and spectrum generating algebras, and, last but not least, on the ability to exploit these ideas in the context of approximate symmetries. Indeed the question of how to define approximate structural invariance and how to relate the many non-classical ensembles used in literature to such approximate invariances may well be the key to further progress along these lines.

References

[1] T. H. Seligman, in: *Proceedings Wigner Symposium,* Goslar 1991 (World Scientific, Singapore, 1992)
F. Leyvraz and T. H. Seligman, *Phys Lett. A* 168, 384 (1992)

[2] c. f. T. A. Brody et. al., *Rev. Mod. Phys.* 53, 385 (1981)

[3] A. I. Schnirelman, *Usp. Mat. Nauk.* 29, 181 (1974)

[4] P. Mello, *Phys. Rev. Lett.* 60, 1089 (1988)
S. Iida, H. A. Weidenmüller and J. A. Zuk, *Ann. Phys. (N.Y.)* 200, 219 (1990)

[5] F. M. Israelev, *Phys. Rep.* 196, 299 (1990)

[6] Y. V. Fyodorov and A. D. Mirlin, *Phys. Rev. Lett.* 67, 2405 (1991)