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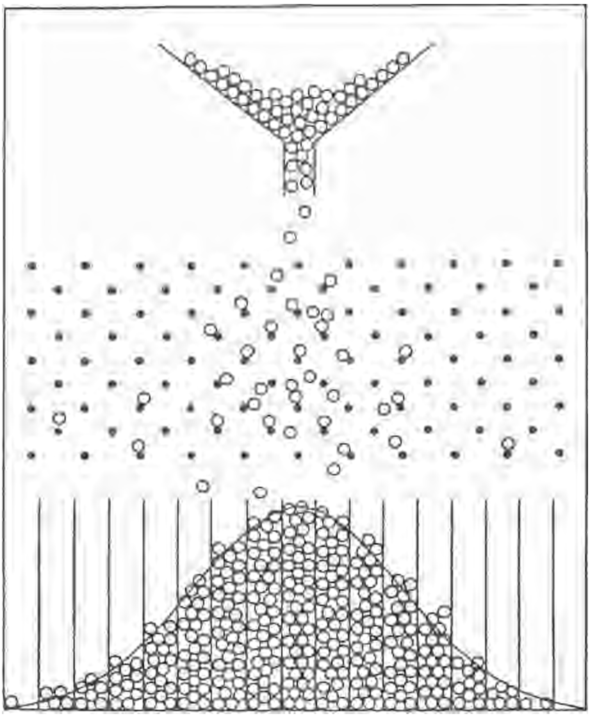
Universality in the Statistical Description of Physical Systems*

The need for a statistical description [1,2]

In order to illustrate the idea, we present in Fig. 1 a setup, known as Galton's board, that we shall now briefly discuss. A marble falling from the top meets a large number of obstacles before it finally ends up in one of the



Hg. 1

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lower containers. At each obstacle, the marble can fall either to the right or to the left of it, so that, after many collisions, the trajectory is indeed a very complex one.

The trajectory is so sensitive to the "initial conditions" that, if we now observe a second marble falling from the top, we find, in general, a different trajectory and a different final container. This is an example of an *unstable trajectory,* for which *small changes* give rise to *large effects.* Thus, if we try to describe the above experiment using the laws of classical mechanics, we have to face the fact that ever longer trajectories become, in the long run, unpredictable. Does this mean that a description of the problem is hopeless? The answer is yes, adding that it will also be uninteresting, if we do not ask an *appropriate question!*

Suppose that, instead of trying to describe each trajectory in detail, we repeat the experiment many times and ask *what fraction of all the marbles ends up in a given container.* If we do that, we observe that a *statistical regularity* emerges: the *distribution* of marbles tends towards being a limiting one, the so called bell-shaped, normal or *Gaussian* distribution sketched in Fig. 1. The lesson we learn is that now we obtain a simple answer! We thus see that a statistical approach to the problem is not just the only feasible one, but, as we shall see further on, it may reveal features which would otherwise remain hidden.

As another illustration, consider the molecules of a gas in equilibrium inside a box: this is an example of a many-body system. In a classical mechanical description, each molecule follows a straight trajectory, until it collides with another molecule or with the walls of the container. The situation is even more complicated than in the first example, because now all the molecules move. Again, the complex, unstable trajectories occurring in this problem are not predictable in the long run, and a statistical approach is indeed more appropriate.

We can thus ask, for instance, what fraction of the molecules can be found in a certain range of velocities $So_x\ c5v\ 80,$. The answer is given by the expression

$$\frac{e^{\frac{mv^z}{2kT}}}{(2\,mkT/m)'^{12}}$$

known as the Maxwell-Boltzmann distribution. In the above equation, m is the mass of each molecule, y the velocity, k a universal constant known as Boltzmann's constant and T the absolute temperature. We see that the answer is a Gaussian distribution centered at zero velocity and with a width which is related to the temperature of the gas. A concept like temperature, indeed a very important one, is revealed to us precisely when a statistical approach is adopted! We also notice that the Maxwell-Boltzmann distribution given above does not depend on the specific interaction between the molecules: i. e., for a given molecular mass and temperature, the distribution is given once and for all! This is an example of a *universal* behaviour, where the only physically *relevant* quantity is the ratio kT/m.

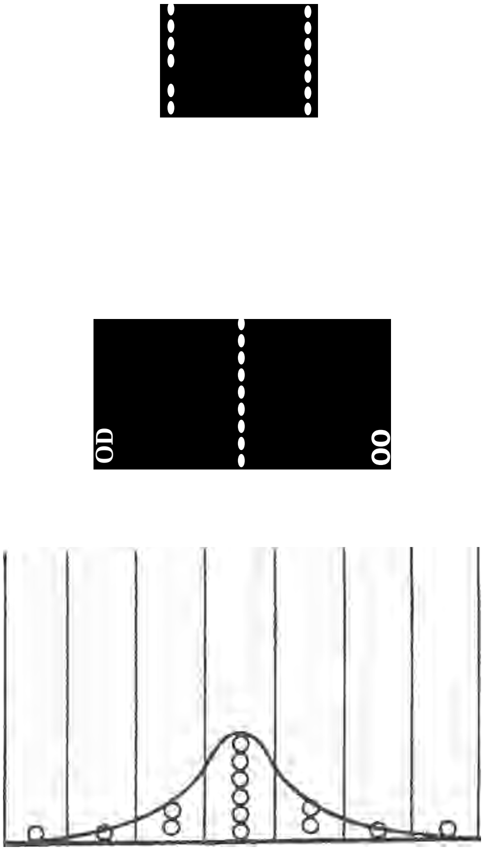

Fig. 2

The central-limit theorem [3]

This is one of the most powerful theorems in the theory of probability. We illustrate it here by means of a simple example. We assign the values 1 and *-1* (indicated below as 1) to the two sides of an "unbiased" coin, that we use to do an experiment.

The first line in the figure indicates that the two sides occur with the same probability when the coin is tossed. If we now consider pairs of outcomes, we have the four possibilities shown in the second line. In the rightmost situation, i. e. 11, the sum of the outcomes is 2; this possibility occurs with the same probability as the leftmost one 1 1, that gives a sum of -2, while the value 0 for the sum occurs twice as frequently. The third line shows the corresponding situation for triplets, and the fourth, for quartets of outcomes. The result gradually becomes similar to a Gaussian distribution! We could interpret the distribution of Fig. 1 as arising from a similar mechanism, where the variable taking on the values 1 and -1 is associated with the right and left displacements of the marble at each obstacle.

A limiting Gaussian distribution is actually approached for an arbitrary initial distribution of the independent variables that are to be added (actually with very mild restrictions)! This is the content of the *central limit theorem,* that was studied by De Moivre, Laplace, Poisson, Herschel, Lyapunov. The resulting Gaussian distribution contains only two parameters, or *relevant quantities:* centroid and width; these depend only upon the centroid and width of the original distribution, other details of the latter being irrelevant! Thus the central limit theorem describes a situation in which a *universal distribution* is attained, and uncovers the relevant quantities.We can also interpret the above situation in a very appealing way"•z.

Suppose that *N* objects are thrown at random in the slots of Fig. 3, with the only requirement that the distribution should have *given centroid and width.* One can show that out of all configurations fulfilling this requirement (three such possibilities are sketched in Fig. 3), *the Gaussian distribution is the one that occurs most frequently!* With a suitable definition of *information,* one can also show that the information carried by a Gaussian is *smallest* among those distributions having the same constraints.

The above observation allows us to rephrase the central limit theorem saying that the sum of a large number of statistically independent random variables (properly normalized; we shall not be more precise here) has a distribution that approaches one of minimum information among those having the same centroid and width! The latter are thus the only relevant quantities left in the problem. We can also interpret the Maxwell-Boltzmann distribution of molecular velocities as the one of minimum information for a given temperature, which is thus the only relevant quantity.

In what follows we shall see how some of the above ideas can be used successfully to study a number of problems in other branches of physics.

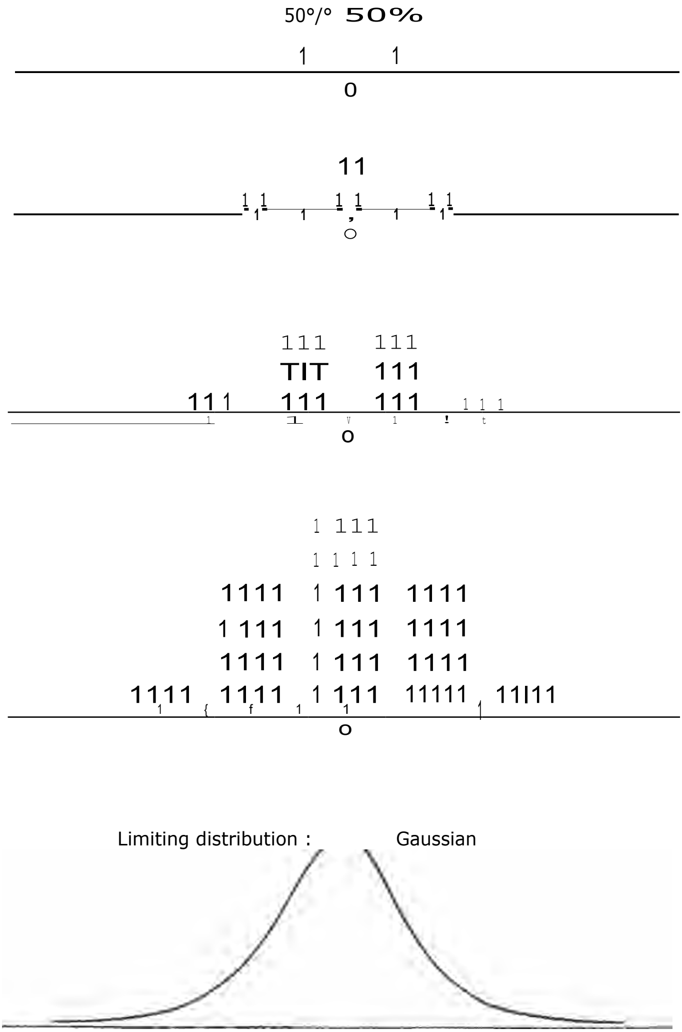

Fig. 3

Scattering problems in physics [1,2]

The oldest way that physicists have to study a system is to throw certain "projectiles" to it and analyze what "comes out": this is called a *scattering* experiment. In everyday life we turn on the light to "see" an object: in fact, what we do is analyze the scattering of light by that object.

In Fig. 4 we present the result of a scattering experiment intended to study an atomic nucleus: the "target" is an isotope of chlorine and the projectiles are protons, accelerated at the energy indicated in the abscissa of the figure: roughly 10 million electron volts. Part of what comes out are a-particles, whose number, detected at a fixed angle, is plotted in the figure.

We observe that the a-particle yield varies with energy in a very complicated way. Actually, nobody knows how to do a calculation, starting from the Schroedinger equation, to reproduce the complex behaviour shown in Fig. 4. But even if one did, one probably wouldn't learn much! Again, just as we saw in the previous sections, a statistical description, which is the only feasible one, may reveal features of considerable physical significance. In a statistical analysis one is interested, for instance, in the plot of Fig. 4 *smoothed, or averaged, over energy,* the size of the *fluctuations* around that average and, more generally, the full statistical distribution of such fluctuations.

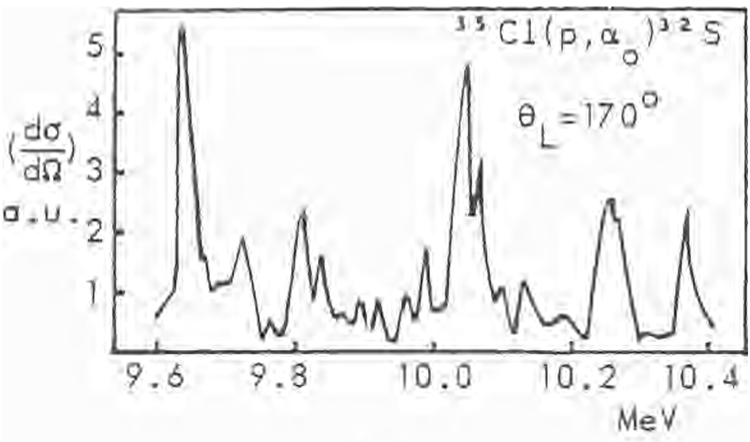

Fig. 4

In a number of problems like the one of Fig. 4 one has found that the *relevant* quantity is the smoothed out behaviour and that, once this is specified, the statistical properties are *universal!* Indeed, the actual statistical distribution carries minimum information among those distributions that have the same smoothed out behaviour! Although the present problem is more complex than the ones considered in the previous sections, and the resulting distributions are, accordingly, more complicated than the Gaussian distributions found there, it is remarkable that a similar philosophy applies.

What is really remarkable is that the notions of relevant quantities, information and universality allow a beautiful description of the statistical properties of a variety of systems whose dimensions differ by so many orders of magnitude. The typical size of an atomic nucleus is ’r 10^{-14}m. In recent years there has been much interest in the electronic conduction in solid state devices called *mesoscopic,* whose typical dimensions are — 10^{-6} – 10^{-7}m; an electron that moves inside such a system sees a random medium, and it is this fact that gives rise to statistical considerations. A similar problem of random scattering is encountered in the study of the propagation of electromagnetic waves in a disordered medium; in the case of microwave experiments one is dealing with macroscopic systems of the order of lm.

We would like to finish with the comment that, in a sense, the guiding philosophy of the above presentation is contained in William of Occam's (1300-1349) famous dictum:

"Essentia non sunt multiplicanda praeter necessitatem",

known as the "Occam Razor". Occam's statement, which, literally, means: "Entities do not have to be multiplied beyond necessity", was rephrased by Bertrand Russell [4] as: "If in a certain science everything can be interpreted without a certain hypothesis, there is no reason to use it".

References:

1 P. A. Mello, "Macroscopic Approach to Statistical Phenomena", *Quantum Chaos and Statistical Nuclear Physics,* Springer Verlag, Berlin, 1986 (Proceedings of the 2nd International Conference on Qantum Chaos, ed. by T. H. Seligman and H. Nishioka). See also references contained therein.

2 P. A. Mello, "Theory of Random Matrices: spectral statistics and scattering problems". Lectures presented at the Miniworkshop on Nonlinearity *Chaos in Mesoscopic Systems,* held at the International Centre for Theoretical Physics, Trieste, Italy, July 1993. See also references contained therein.

3 M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics,* Griffin, London, 1952, p. 193.

4 B. Russell, *A History of Western Philosophy,* (New York, 1945 / London, 1946) vol. II, part III, ch. VIII.