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# On Constitutive Functionals

Es werden hier eindimensionale Materialgleichungen mit Gedächtnis betrachtet, in denen die Spannung als Funktional der Dehnungsgeschichte angesehen wird. Wir unterscheiden zwischen Stoffen mit instantaner Elastizität und Stoffen mit instantaner Viskosität. Die relevante Stetigkeitsbedingung im ersten Fall lautet, daß das Spannungsfunktional im Sinne der Supremum Norm stetig von der Dehnungsgeschichte abhängt. Im zweiten Fall macht man besser die stärkere Annahme, daß das Spannungsfunktional von der Geschichte der Dehnungsänderung stetig abhängt - wieder im Sinne der Supremum Norm. Es wird diskutiert, in welcher Weise eine nichtlineare funktionale Abhängigkeit der Spannung von der Geschichte der Dehnung oder Dehnungsänderung angenähert werden kann durch ein Polynomfunktional, und verschiedene Methoden der Darstellung solcher Polynomfunktionale werden behandelt.

## 1. Introduction

In 1957, Green and Rivlin [1] discussed the restrictions which must be satisfied by a constitutive equation for isothermal deformations of a material with memory. They made the constitutive assumption that the Cauchy stress matrix at a time *t* depends on the history of the deformation gradient matrix up to and including time *t, i.* e. that it is a matrix-valued *functional of* the history of the deformation gradient matrix in some time interval $[t_o, t]$. They assumed that this dependence is continuous in the sense of the supremum norm. They imposed on the constitutive functional the restrictions implied by the *assumption* that the superposition on the assumed deformation of an arbitrary time-dependent rigid rotation causes the stress matrix to be rotated by the amount of this rotation at time *t.* This assumption has become known, inappropriately, as the Principle of Material Frame Indifference. In [ 1] Green and Rivlin also discussed the manner in which the further restrictions due to any symmetry the material may possess can be imposed on the constitutive functional. This approach to the theory of constitutive equations has been developed and extended by Rivlin and various collaborators.

In [1] and in later papers the explicit representation of constitutive functionals is also discussed. It is this aspect of constitutive equation theory which is discussed in the present paper. In order to highlight the essential elements of the problem, this is done within the framework of one-dimensional constitutive equations.

Attention is first drawn to the fact that different constitutive assumptions are appropriate to the discussion of materials which possess instantaneous elasticity and those which possess instantaneous viscosity. While in the former case the stress is assumed to depend on the history of the strain, in the latter it is assumed to depend on the history of the rate of change of strain. In either case the stress is assumed to be a continuous functional of the argument function in the sense of the supremum norm. It is argued that this is a necessary condition for meaningful experiments on the material considered to be possible.

In order to accommodate strain histories, or rate of change of strain histories, extending into the infinite past it is necessary that the constitutive functional be convergent as the time interval [$t_o$, *t]* tends to (— co, *t]*. It is shown how for representations of the constitutive functional as a series of multiple integrals this requirement places restrictions on the kernels in the integrals.

We note that the continuity assumption made in the present paper differs from that made by Coleman and Noll [2] in their much advertised "Principle of Fading Memory". They assume continuity of the constitutive functional in the sense of a weighted Hilbert norm, in which the weighting factor depends on the particular material considered. They claim that convergence of the constitutive functional for histories extending to the infinite past is ensured by placing certain restrictions on this weighting factor. It is not clear that a condition of this kind, or indeed any assumed continuity condition, can provide appropriate conditions for the constitutive functional to be convergent as $t_o$ — — co. For, it does not appear possible to give meaning to a continuity condition on a functional of a strain, or rate of change of strain, history extending to the infinite past without first assuming that these functionals are convergent. Moreover, it does not seem, except perhaps in certain particular cases, that the assumption of continuity in the sense proposed by Coleman and Noll can provide explicit limitations on the physical parameters occurring in a constitutive equation of specified form.

In §§ 8-12 we discuss, still in the context of one-dimensional constitutive relations, the explicit representation of nonlinear constitutive functionals. It is seen that if the constitutive functional is polynomial it can be represented by a sum of multiple integrals. Various ways in which a non-polynomial functional can be approximated by a polynomial functional are demonstrated.

While the discussion in this paper is restricted to one-dimensional constitutive relations, the methods can easily be applied to give corresponding results for tensorial constitutive relations applicable to three-dimensional deformations. To some extent this has already been done in earlier papers (see, for example, [3]). To the extent that it has not, it will be done in a later paper.

## 2. Materials with memory possessing instantaneous elasticity

Suppose a thin rod to be loaded with a time-dependent load *H (t)* per unit initial cross-sectional area. Let *e (t)* be the corresponding fractional extension (extension at time *t*/ initial length). If the material of the rod is elastic

$$17\ (t) = E\ e\ (t), \qquad (2.1)$$

where *E* is a constant - Young's modulus. For a viscoelastic material, or a material with memory, *H (t)* depends on the history of *e (r)* up to time *t.* We suppose that prior to time $t_o$, the rod is in equilibrium under zero load, so that

$$e\ (r) = 0,\ \ H\ (t) = 0\ T, t < t_o. \qquad (2.2)$$

We may write the constitutive assumption for *H (t)* as

$$17\ (t) = U\ \{E\ (T)\}\quad T\ E\ [t0, t], \qquad (2.3)$$

where $_{t8}$ denotes some functional of the argument function *e* **(T).**

We will suppose in this section that the material of the rod possesses instantaneous elasticity. In this case we suppose that the strain histories *a (r)* lie in a function space, $CA_P$ say, which is the space of functions of bounded variation, defined for *r* E [to, *t],* which are piece-wise continuous or possess at most a countable number of jumps.

A is assumed to be a continuous functional of its argument function in the sense of the supremum norm; i. e. if $e_1$ (r) and $e_2$ (r) are two strain histories in $_{Oe}$ and *H (t)* and **H2** *(t)* are the corresponding stresses at time *t,* then

$$H\ (t) - 17_2\ (t)\ I \ {-+}\ 0 \text{ as sup } I\ E1\ (T) - e2\ (T)\ I \ {-+}\ 0 \qquad T\ E\ [to, t]. \qquad (2.4)$$

We note that two strain histories $6_1$ *(r)* and $e_2$ (r) which are such that

$$\sup I\ e_1\ (r) - e2(T)\ I = 0 \qquad T\ E\ [r0, t] \qquad (2.5)$$

are indistinguishable experimentally. Consequently, if the assumption (2.4) were not satisfied we would have, in the limit, $17_1$ *(t)* * $17_2$ *(t); i.* e. two experimentally indistinguishable strain histories would lead to different values for the stress and, accordingly, the experiment would not be meaningful as we would not know to which strain history the measured stress corresponds.

The functional a must necessarily also satisfy the condition that *17 (t)* be convergent for $t_o$ - co. Otherwise infinite stress may be associated with strain histories which are bounded but extend into the infinite past.

## 3. Linear instantaneously elastic materials

We now restrict our discussion to the case when A is a linear functional * and to emphasize this rewrite (2.3) as

17 (t) = {E (r)), e (r) E $e_e$, T E [$t_0$, t]. (3.1)

We consider first a deformation history for which the strain is zero up to some time T* in the interval [$t_o$, t] and is thereafter unity, i. e.

E (r) = H (r — r*), (3.2)

where H (r — T*) is the Heaviside unit step function defined by

H(r—r*)= 1 TAT* / 0 r<r . (3.3)

Let g (r*, t) be the value of H (t) associated with this deformation history. Then, from (3.1)

g(r*,t)He(r—r*)}. (3.4)

We now consider an arbitrary piece-wise continuous strain history a (r), which has at most a finite number of jumps. We divide the time interval [$t_o$, t] into n subintervals [$T_o$, $T_i$], ($T_i$, $T_2$], . . . , ($r_{n-1}$, $r_n$], where $T_o$ = $t_o$, $T_n$ = t. If E (r) has jumps the intervals are chosen so that these occur at one or more of the times $r_1$, ... , $T_n$.

Let (r) be the strain history given by

E (r) _ [e (T;) — e (ri 1)] H (r — $r_e$). (3.5)
i=1

The corresponding stress H(t) is given by

ft (t) _ { (r)}, (3.6)

Since a is a linear functional, (3.4) and (3.5) yield

H (t) = [a (ri) — E (r;–i )~ g (r,, t). (3.7)

We now take the limit of (3.7) as n co and sup ($r_i$ — Ti_ 1) —+ 0 to obtain, with (3.6), an integral representation for H (t):

H (t) = g(r, t) de (r). (3.8)

to

If the material is of the hereditary type then (3.8) may be rewritten in the form

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* This means that for any two functions $e_1$ (r) and $e_2$ (r) in $e_e$

2 lei (r) +$e_2$ (r)} =$ {$e_1$ (r)} +2 {$e_2$ (r)}

and E is a continuous functional of e (r).

$$17\,(t) = g\,(t - z)\,d\,(z). \quad t_o \tag{3.9}$$

For strain histories *E* (r) which are differentiable with respect to time, we can rewrite (3.8) as

$$17\,(t) = g(r, \quad (z)\,d\,z, \quad t_o \tag{3.10}$$

where the dot denotes differentiation with respect to time. Equations (3.8) and (3.9) may also be rewritten as

$$17\,(t) = g(t, t)\,E\,(t) - e\,(z)\,dg\,(z, t) \quad t_o \tag{3.11}$$

and

$$17\,(t) = g(^0)\,e(t) - E\,(r)\,dg(t - z) \tag{3.12}$$

respectively.

## 4. Linear non-instantaneously elastic materials

In obtaining the result (3.8), it was assumed that the material considered has instantaneous elasticity, so that the strain histories *H (r* — r*) and EH are physically possible ones. *If* the material does not possess instantaneous elasticity we can invoke the Hahn-Banach theorem to carry through the proof. It follows from this theorem that if E {E (r) } in (3.1) is a linear functional of *E* (r) for *e (z)* continuous and of bounded variation, we can construct a functional, 2 {e (r) } say, which is a linear functional of *E* (r), for E (r) of bounded variation and possessing at most a countable number of jumps, such that

$$(T)1 = 2\,\{e\,(T)1 \tag{4.1}$$

for *E* (r) continuous and of bounded variation. Then from (3.8) it follows that E {e (r) } must be expressible in the form

$$\{E\,(z)\} = g\,(z, t)\,dE\,(z). \quad t_o \tag{4.2}$$

Hence, from (4.1) and (3.1) it follows that 17 *(t)* must be expressible in the form

$$H(t) = \int_{t_0} g(\tau, t)\, de(\tau). \tag{4.3}$$

The relation (4.2) is, of course, valid for $e(\tau)$ possessing at most a countable number of jumps, while (4.3) is valid only if $e(\tau)$ is a continuous function.

In the case discussed in § 3, in which the material considered has instantaneous elasticity, the kernel $g(\tau^*, t)$ has a clear physical interpretation as the stress at time $t$ which results from a strain history of the form (3.2). This is not the case if the material does not possess instantaneous elasticity, since the strain history (3.2) is not a physically possible one. However, a physical interpretation, albeit less explicit, can be given to $g(\tau^*, t)$ in a variety of ways. For example, we may consider a deformation in which the strain increases linearly in the time interval $[t_o, \tau^*]$ and then remains constant:

$$e(\tau) = \begin{cases} x(\tau - t_o) & t_0 < \tau < \tau^* \\ x(\tau^* - t_o) & \tau^* < \tau < t. \end{cases} \tag{4.4}$$

Then (4.3) yields

$$H(t) = x \int_{t_0}^{\tau^*} g(\tau, t)\, d\tau. \tag{4.5}$$

From this we obtain

$$g(\tau^*, t) = \frac{1}{x} \frac{dH(t)}{d\tau^*} \tag{4.6}$$

## 5. Materials with memory possessing instantaneous viscosity

In this section we suppose that the material considered does not possess instantaneous elasticity, but may possess instantaneous viscosity; i. e. a jump in the rate of change of strain may result in a non-infinite change in the stress.

As in § 2 we suppose that $e(\tau) = 0$ for $\tau < t_o$, but now consider that the strain histories are differentiable with respect to time. We replace the constitutive assumption (2.3) by

$$H(t) = \mathcal{F}\{\dot{e}(\tau)\} \quad \tau \in [t_0, t], \tag{5.1}$$

where the dot denotes differentiation with respect to $\tau$; the stress is assumed to be a functional of the rate of change of strain rather than of the strain.

We suppose that $\dot{e}(\tau)$ lies in a space, $\mathcal{E}$, say, of bounded variation for

T E [to, T] and is piece-wise continuous or has at most a countable number of jumps. is assumed to be a continuous functional of é Win the sense of the supremum norm; i. e. if a (r) and $a_2$ (T) are two strain histories and $17_1$ *(t)* and 172 *(t)* are the corresponding stresses at time *t*, then

$$| Hi (t) - n2 (t) | \quad 0 \text{ as sup } | e1 (r) — a_2 (T) | \quad o, \qquad T \; E \; [t_0, t]. \tag{5.2}$$

Provided that *a* (r) is not discontinuous at T = $t_o$ and r = *t* the strain history *â* (r), together with the initial condition *e* $(t_o)$ = 0, determines the strain history *e* (T). It follows that the constitutive assumption (5.1) is equivalent to a constitutive assumption of the form (2.3); i. e.

$$17 (t) = \quad (r) \} = A \{e (T)\} \qquad T \; E \; [to, t]. \tag{5.3}$$

However, the assumption that $_A$ is a continuous functional of *e (r)* does not imply that is a continuous functional of *a* (r). Continuity of $_A$ implies that for any a > 0 there exists a quantity *6 (a)* > 0 such that

$$|A^{\{e1} (T) \} — O \{e2 ("0) | < a, \tag{5.4}$$

if

$$\sup | e1 (T) — e2 (T) | < 6 (a) \qquad r \; E \; [to, t]. \tag{5.5}$$

Similarly, and equivalently to (5.2), continuity of g implies that for any a > 0 there exists a quantity 3(a) > 0 such that

$$g \quad (T) | — \{i2 (T) \} | < a, \tag{5.6}$$

if

$$\sup | e_1 (r) — e_2 (r) | < b (a) \qquad T \; E \; [t0, t]. \tag{5.7}$$

While, in view of (5.3), the relations (5.4) and (5.6) are the same, the relation (5.5) does not imply (5.7). However, *(5.7)* does imply (5.5). To see this we note that

$$e1 (T) - e2 (T) = \int [e1 (O - i2(O] \, dS$$

$$< \sup | E1 (\sim) - e2 (S) | (t - t0), \qquad E \; [t0, t]. \tag{5.8}$$

Then by taking

$$6 (a) = (t — t_o) \, 8 (a), \tag{5.9}$$

we obtain (5.5) from (5.7).

If *a* (r) may be discontinuous at T = $t_o$ or r = *t*, then the constitutive assumption (5.1) is equivalent to a constitutive assumption of the form

$$17 (t) = \underset{6 \quad 0}{Lt} A \{e (T)\} \qquad T \; E \; [t — S, t + 8], \tag{5.10}$$

with *6* > 0.

# 6. Linear instantaneously viscous materials

We now suppose that tg in (5.1) is a linear functional and to emphasize this fact we write

17 (t) = (T) } T E $[t_o, t]$. (6.1)

Paralleling the argument in § 3 we obtain a representation for 17 (t) in the form (cf. (3.8))

17(t) = h(T, t) dr (r). (6.2)

to

If the material is of the hereditary type, (6.2) may be written in the form

*II* (t) = h (t — r) d (T). (6.3)

$t_o$

With the assumption that (r) = 0 for r < $t_o$, (6.2) and (6.3) can be rewritten as

17(t) = h (t, t) (t) —J ~ (T) d h (T, t) (6.4)

$t_o$

and

t

(

17 (t) = h (0) (t) — é (T) d h (t — r). (6.5)

$t_o$

With the assumption that *h* (r, t) is differentiable with respect to T, we can rewrite (6.4) as

r

(

17 (t) = h (t, t) É (t) — J h (r, t) é (T) dr. (6.6)

Equation (3.10) can be recovered from (6.6) by taking

h (t, t) = 0, h (T, t) = — g (T, *t*). (6.7)

It is evident from (6.2), as from the argument leading to it, that *h* (T, t) can be interpreted physically as the stress at time t resulting from a strain history given by

e(T) _ 0 TG T
T z < z < *t*, (6.8)

so that

(T) _ TGT
1 i < T < t. (6.9)

# 7. Strain histories extending to the infinite past - linear materials

We have noted in § 2 that the functional in (2.3) must necessarily converge for $t_o$ - oo. This must evidently also be true for the functional g in (5.1). Accordingly, the linear functionals e and Ê in (3.1) and (6.1) respectively must converge for $t_o$ -+ - oo. This fact places restrictions on the kernels *g (r, t)* and *h (r, t)* in the integral representations (3.8) and (6.2).

We have seen in § 3 that for the strain history given by (3.2)

$$n(t) = g(r^*, t). \tag{7.1}$$

It follows that a necessary condition for convergence of the integral in (3.8) as $t_o$ –. - co is that *g (r, t)* be bounded for r E (- co, *t].* This condition is also sufficient provided that *e* (r) has bounded variation in (- co, *t], i.* e. provided that *I de (r) I* converges. To see this we observe that

$$\int^{r} g(r, t)\, dc(r)1 <. \int^{r} I\, g(r, t)\, I\, I\, (Jr(T)\, I$$

$$MJ \int de(r)1, \tag{7.2}$$

where *M* is the upper bound on 1 *g (r, t)* |.

If *a* (r) is bounded in (- co, *t],* has bounded variation in every finite interval [$t_o$, *t],* but does not have bounded variation in (- co, *t],* then a somewhat stronger condition on *g (r, t)* is required in order to ensure convergence of the integral in (3.8). Such a condition is that *g (r, t)* have bounded variation in (- co, *t], i.* e. that the integral $_1$ I *dg (r, t) I* converges. To see this we note that (cf. (3.11))

$$\int_{co} g(r, t)\, de(r) = g(t, t)\, e(t) - \int_{-CO} e(r)\, dg(r, t). \tag{7.3}$$

Let *M* be the upper bound of I e (r) I in the interval (- co, *t].*
Then

$$\int^{r} e(r)\, dg(r, t)\ 5 \int^{r} (r)I\, I\, dg(r, t)I$$

$$< M \quad I\, dg(r, t) I \tag{7.4}$$

and accordingly the integrals on the right and left hand sides of (7.3) both converge.

Restrictions analogous to those given above evidently apply *mutatis mutandis* to the kernel *h (r, t)* in (6.2).

## 8. Polynomial functionals

Let Q3 *{f 1(x), f2*(x) } be a functional of two functions f *(x), f2(x)* with *x* E *[a, b]* for all functions lying in a specified function space, e say. For brevity we shall also denote the functional by C3 (, f2 }. C3 is said to be *bilinear* in~f and j; if Q3 is a continuous functional of f, and of f2, and

V1') +. 1²),.f2 } = V1">f2 } + ~3 V1²),.f2 },

Ci if, , +j2> All } + C³ {i;,.4 } (8.1)

for all f and f2 in the function space Chi.

We define analogously a *multilinearfunctional Ci { f (x), ... ,f (x) }* of the functions f *(x), ... ,f p(x)* with *x* E *[a, b]*, for all functions lying in the specified function space '. For brevity we denote the functional by *,f }. G3* is a continuous functional off*a* and is said to be multilinear in f1, ... , *f* .. if

= 3 {f,... ,f(a'),...,f„} + 43 {f,...,f(²),...,f,1} (8.2)

for all a and for all functions lying in ' .

From any multilinear functional 03 {/ , ... , f, } we can form a multilinear functional which is symmetric with respect to permutation off, , ... ,f, . We denote this by A { f , . . . ,f, }:

(8.3)

where the summation is over all permutations off , ... , *fa* .

We now choose as the function space e the space of functions of bounded variation which are piece-wise continuous or have at most a countable number of jumps. Since A is a linear functional of each of the argument functions f , ... ,f,, in the sense of the supremum norm, we can obtain a representation for A as a multiple integral thus:

$$\Phi_{\mu}\{f_1(x),\ldots,f_p(x)\} = \int_a^b \cdots \int_a^b g_\mu(x_1,\ldots,x_p)\, df_1(x_1)\ldots df_\mu(x_\mu). \tag{8.4}$$

In (8.4) we now take

$$f_1(x) = f_2(x) = \ldots = f_p(x) = f(x) \text{ say.} \tag{8.5}$$

We obtain

$$\Phi_\mu\{\underbrace{f(x),\ldots,f(x)}_{\mu \text{ arguments}}\} = \int_a^b \cdots \int_a^b g_\mu(x_1,\ldots,x_p)\, df(x_1)\ldots df(x_p), \tag{8.6}$$

with $g_p$ symmetric under interchange of $x_1, \ldots, x_p$.

$\{f(x), \ldots, f(x)\}$ is said to be a *homogeneous form of degree lc* in the function *f(x)*. (We note that if,u = 0 in (8.6) then A is a constant.) The sum of y + 1 such forms of degrees 0, 1, ... , y is called a *polynomial functional* of degree v. It is evident that a polynomial functional defined in this manner is a continuous functional off *(x)* for all *f(x)* in the function space ▪

# 9. Strain histories extending to the infinite past - nonlinear materials

We now return to the rod considered in § 2 and assume that in (2.3) A is a polynomial functional of degree v. Then, from the result of the previous section it follows that *H* (t) may be expressed as the sum of y multiple integrals thus:

$$\Pi(t) = \sum_{a=1}^{v} I_a, \tag{9.1}$$

where

$$I_a = \int_{t_0}^{t} \cdots \int_{t_0}^{t} g_a(\tau_1,\ldots,\tau_a, t)\, dE(\tau_1)\ldots dE(\tau_a), \tag{9.2}$$

and $g_a$ is symmetric under permutation of $\tau_1, \ldots, \tau_a$. We note that since $\Pi(t_0) = 0$, the (constant) term corresponding to a = 0 becomes zero.

Since *17(t)* must be finite for strain histories extending to the infinite past, it follows that each of the integrals in (9.1) must converge for to — 03. We consider the conditions on $g_a(\tau_1, \ldots, \tau_a, t)$ for $I_a$ to converge in the case when a = 2. The conditions for arbitrary a will then be evident.

We consider a strain history *e*(r) defined by

$$e(\mathrm{r}) = c_1 H(r - \quad + c_2 H(r - \mathrm{r}2), \tag{9.3}$$

where $c_1$ and $c_2$ are constants. Then from (9.2), $1_2$ is given by

$$1_2 = c1\, g2\,(4, \mathbf{t}\, , t) + c2\, g2\,(T2, T2\, t)$$
$$+ 2\, c_1\, c_2\, g2(\mathrm{ri}, 4, \mathrm{t}). \tag{9.4}$$

It follows that a necessary condition for convergence of $/_2$ as $t_o$-. - co is that $g_2$ ($r_i$, $r_2$, *t)* be bounded for $r_1 < t$ and $r2 < t$. This condition is also sufficient provided that *a* (r) has bounded variation in (- oo, *t],* since (cf. (7.2))

$$\mathrm{J} \quad \mathrm{J} \quad g(z_1, z_2, t)\, de(z_1)\, de\, (TO$$

t t

$$\leq \mathrm{J} \quad \mathrm{J}\ \mathrm{Ig}(z^1, \mathrm{T2}, \mathrm{t})\mathrm{I}\ \mathrm{Ide}(z1)\mathrm{I}\ \mathrm{I}\ de\,(\mathrm{T2})\mathrm{I}$$

t t

$$< M \quad \mathrm{J} \quad \mathrm{J}\ \mathrm{I}\ de\,(\mathrm{r1})1 \quad \mathrm{l}\ de\,(\mathrm{T2})1, \tag{9.5}$$

where $M$ is the upper bound on $g\,(z_1, r_2, t)$ for $r_i < t$ and $r_2 < t$.

It is evident that analogous conditions for the convergence of $'_a$ as $t_o$-. - co are valid, for arbitrary a. A necessary condition for convergence of $1_a$ as $t_o$ -+ - co is that $g_a$ ($r_{17}$ ... , $r_{Q)}$ t) be bounded for $r_ß < t$ *(ß* = 1, ... , a). This condition is also sufficient provided that *e* (r) has bounded variation in (- co, *t].*

## 10. Non-polynomial functionals

Let $rp_.(x)$ *(i* = 1, 2, ...) be a specified set of linearly independent functions, defined for $x$ E *[a, b].* Let **e** be the space of functions defined by the uniformly convergent series

$$f(x) \quad \sum_{i=1}^{co} c_i\, 9_i(x), \tag{10.1}$$

where the c's are real numbers. Then the functions $sp_i\,(x)$ lie in the space 0, as do the partial sums $f_n(x)$ defined by

$$fn(x)\ \_ \sum_{i=1}^{n} c\ IP_,(x). \tag{10.2}$$

Since iff *(x)* is specified $c_i$ *(i* = 1, 2, ...) are determined uniquely, *c* is a functional of *f (x);* it is evidently a linear functional. Conversely, if

$c_i$ ($i$ = 1, 2, ...) are specified, $f(x)$ is uniquely determined. Accordingly, for each value of $x$, $f(x)$ may be regarded as a *function* of the countable set $c_i$ ($i$ =1, 2, ...). It follows that any functional 8 off $(x)$ may be expressed as a function of $c_i$ ($i$ = 1, 2, ...):

$$A\,\mathrm{If}\,(x)\,\} = F(c_1, c_2, \ldots). \qquad (10.3)$$

We shall consider functionals 8 which are continuous functionals off $(x)$ in the sense of the supremum norm.

Since from (10.1), for each $x$, the dependence off $(x)$ on each $c_i$ is continuous, it follows that if is a continuous functional off $(x)$, then $F$, defmed by (10.3), is a continuous function of each of the quantities $c$

*Example 1*

Let

$$[a, b] = [0, b], \quad \varphi_i(x) = \sin\frac{2\pi ix}{b}. \qquad (10.4)$$

Then, $f(x)$ is given by the Fourier series

$$f(x) = \sum_{i=i} c_i \sin\frac{2\pi i x}{b} \qquad (10.5)$$

$c$ is the functional off $(x)$ defined by

$$c_i = \frac{2}{b}\int_o^b f(x)\sin\frac{2\pi ix}{b}\,dx \qquad (10.6)$$

and 8 may be regarded as a function of the quantities $c_i$ ($i$ = 1, 2, ...) so defined.

*Example* 2

Let $f(x)$ be the functions expressible as Taylor series about $b$; so that

$$\varphi\,(x) = (x - b)^{i-1}. \qquad (10.7)$$

Then,

$$f(x) \quad \sum_{i=i} c\,(x-b)^{i-1}, \qquad (10.8)$$

where

$$c = \frac{1}{(i-1)!}\,\frac{d^{i-1} f(x)}{dx^{i-1}}\bigg|_{x=b} = \frac{1}{(i-1)!} f^{(i-1)}(b),\ \text{say} \qquad (10.9)$$

and 8 may be regarded as a function off $^{(i-1)}(b)$ ($i$ = 1, 2, ...).

Since 𝕋 is by assumption a continuous functional off $(x)$,

$$18\{f(x)\} - 8(f_n(x))\,| \to 0 \text{ assupI } f(x) - f_n(x)| \to 0. \qquad (10.10)$$

Also, from (10.1) and (10.2)

sup I $f$ (x) — $f_n$(x) I 0 as n —. co. (10.11)

It follows that

$I$ {$f$ *(x)*} —j ($f_n$ *(x)*}I —.Oas $n$ (10.12)

Then with (10.3) we obtain

I8 {f (x)}— $F_n$(c$_i$,..., e$_n$) I —. 0 as $n$ (10.13)

where

$F_n$(c$_1$,..., c,) = F(c$_{1i}$ ..., c$_{n,}$ 0,0,...). (10.14)

Since $F_n$ is a continuous function of $c_1$, ..., $c_n$, we can, from Weierstrass's Theorem, express it with any desired accuracy by a polynomial in $c_{1i}$ ... , $c_n$; i. e. for any a > 0 we can construct a polynomial Pnm) ($c_1$, ... , c,) say, of total degree m (a), such that

| $Fn$ *(c1,* ... , cn) — Pnm) (c":.. , cn) I < a. (10.15)

From (10.13) it follows that for any *ß* > 0, there exists an integer, $n_o$(ß) say, such that

I8{f(x)}— $F_n$(c$_1$,...,c$_n$]I <ß (10.16)

for $n$ > $n_0$ *(ß).* From (10.15) and (10.16) we obtain

13 { (x) } — Pnm) (c$_i$, ... , Cn) I

I8 i (x)}—Fn(C1, •••,cn)

\+ IF$_n$ C$_1$,...,C$_n$) — Pn$^{(m)}$(c1~...,Cn) |

< a + ß. (10.17)

Thus, by choosing $n$ large enough we can construct a polynomial in $c_{1i}$ ... , $c_n$, which approximates j {f *(x)*} with an error less than a + ß, i. e. with any specified accuracy. Since $c_1$, ... , $c_n$, are linear functionals off *(x),* *Pnm)* ($c_1$, ... , $c_n$) is also a functional off *(x),* and from the definition of a polynomial functional given in § 8 it is a polynomial functional.

## 11. A further polynomial functional approximation

We suppose thatf *(x)* lies in a space, say, of functions of bounded variation, defined for *x* E *[a, b],* which are piece-wise continuous and have at most a countable number of jumps. Let A { f (x) } be a functional off *(x)* which is continuous in the sense of the supremum norm. It will be shown in this section that 6 may be approximated with any desired accuracy by a polynomial functional off (x).

As in § 3 we divide the interval $[a, b]$ into $n$ subintervals $[x_{i-1}, x_i]$ $(i = 1, \ldots, n)$, where $x_0 = a$, $x_n = b$.

We define the function $f_n(x)$ by

$$f_n(x) = \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\, H(x - x_r) + f(x_0), \tag{11.1}$$

where $H()$ denotes the Heaviside unit step function. $f_n(x)$ is evidently a continuous function of each of the quantities $f(x_i)$. We now let $n \to \infty$ while $\sup (x_i - x_{i-1}) \to 0$. Then,

$$f(x) = \operatorname*{Lt}_{n \to \infty} f_n(x). \tag{11.2}$$

Since $\mathfrak{F}$ is a continuous functional, it follows that

$$|\mathfrak{F}\{f(x)\} - \mathfrak{F}\{f_n(x)\}| \to 0 \text{ as } n \to \infty. \tag{11.3}$$

Since $f(x_i)$ $(i = 0, \ldots, n)$ determine $f_n(x)$ uniquely, it follows that $\mathfrak{F}\{f_n(x)\}$ may be regarded as a function, $F_n$ say, of $f(x_0), \ldots, f(x_n)$:

$$\mathfrak{F}\{f_n(x)\} = F_n[f(x_0), \ldots, f(x_n)]. \tag{11.4}$$

Since $\mathfrak{F}$ is a continuous functional of $f_n(x)$, $F_n$ is a continuous function of each of its arguments $f(x_i)$. Paralleling the argument in § 10 it follows from (11.3) and (11.4) that for any $\alpha > 0$ there exists an integer, $n_0(\alpha)$ say, such that

$$|\mathfrak{F}\{f(x)\} - F_n[f(x_0), \ldots, f(x_n)]| < \alpha \tag{11.5}$$

for all $n > n_0(\alpha)$.

Since $F_n$ is a continuous function of its arguments it follows from Weierstrass's Theorem that we can approximate it with any desired accuracy, $\beta$ say, by a polynomial, $P_n^{(m)}$ say, of total degree $m(\alpha)$ in $f(x_0), \ldots, f(x_n)$. With (11.5) it follows that

$$|\mathfrak{F}\{f(x)\} - P_n^{(m)}[f(x_0), \ldots, f(x_n)]| < \alpha + \beta; \tag{11.6}$$

i. e. we can approximate the functional $\mathfrak{F}\{f(x)\}$ with any desired accuracy by a polynomial in the values of $f(x)$ at $x = x_0, \ldots, x_n$, if $n$ is large enough. $f(x_i)$ $(i = 0, \ldots, n)$ are evidently linear functionals of $f(x)$. Accordingly, $P_n^{(m)}$ is a polynomial functional of $f(x)$. We conclude that $\mathfrak{F}\{f(x)\}$ may be approximated with any desired accuracy by a polynomial functional of $f(x)$.

## 12. Fréchet differentiable functionals

A functional A if (x) } is said to be *n* times Fréchet differentiable in the sense of the supremum norm at a function *f (x)* if

to { f (x) +:*i(x) 1* = to {f (x) } + i=1 ) ~r *{f (x) }*

+ ß„ {f (x) }, (12.1)

where 0, { f (x)1 is a homogeneous form in f *(x)* of degree *i* and ß,,{ f (x)1 0 as sup I f (x) I -+ 0 faster than [sup I f *(x) I ] n.* Both ei and $ß_n$ depend on the choice of the function *f (x), i.* e. are functionals of *f (x).*

*i { f (x) }* is called the ith. Fréchet differential (or strong differential) of A if (x) } at the function *f (x).*

If *f (x)* = 0, we obtain from (12.1)

A {f (x) 1= 1/ (x) 1 + ßn if *(x)* }, (12.2)

where $ti_n$ is the polynomial functional of degree *n in f (z)* defined by

$n (f (x) } = A {0 } + 3*it (x) }* (12.3)

The assumption that A If *(x) 1* is *n* times Fréchet differentiable at *f (x)* = 0 is thus nothing more than the assumption that it can be approximated by a polynomial functional off *(x)* of degree *n,* with an error which tends to zero as sup I *f (x) I* —. 0 faster than [sup I *f (x) j] n*

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